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FUNCTIONS: NOTATION AND TERMINOLOGY

Contents

Terminology for functions

Other words meaning "function"

Functions are called by many other names in the literature. This section mentions some of the most common words used to mean "function". ***Authors may not explain the terminology they use.***

See No standardization.

Map, mapping

A function is very commonly referred to as a **map** or **mapping**. In some cases, an author will impose conditions on a function to be called a "map". For example, some require that a mapping be a *continuous* function. Another usage is that a map have a specified codomain. Other uses of the word are described in Wikipedia.

The names "map" and "transformation" are suggested by common metaphors for functions.

Transformation

The word is commonly used when the domain and the codomain are the same (as in a transformation group), but that is not the case when it is used in the phrase linear transformation.

Operator

A function between vector spaces may be called an **operator**. Most functions called "operator" are linear, and the word seems to be most commonly used when the domain is a function space. For example, taking the derivative is a linear operator on some suitable vector space of differentiable functions. The article in Wikipedia gives an extensive list of examples.

Functional

The word **functional** is used as a noun to denote some special class of functions. The most common use seems to be to denote a linear function whose domain consists of some vector space and whose values are elements of the field. But the word is used in other senses, as well. See the Wikipedia entry.

In mathematical English, the word "functional" is a ***noun***. In ordinary English, it is almost always an adjective whose meaning has ***little to do with the mathematical meaning***.

Operation

A function of the form $f : S \times S \to S$ may be called a **binary operation** on S. The main point to notice is that it takes pairs of elements of S to the ***same set*** S.

See cartesian product.

A binary operation is a special case of **n-ary operation** for any natural number n, which is a function of the form $f : S^n \to S$. A 1-ary (unary) operation on S is a function from a set to itself (such as the map that takes an element of a group to its inverse), and a 0-ary operation on S is a constant.

It is useful at times to consider **multisorted algebra**, where a binary operation can be a function $f : S_1 \times S_2 \to S_3$ where the S_i are possibly *different* sets. Then a unary operation is simply a function.

In the 1960's some mathematicians (not algebraists) were taken aback by the idea that addition of real numbers (for example) is a function. I observed this personally. I don't think any mathematician would react this way today.

Calling a function a multisorted unary operation suggest a different way of thinking about it, but as far as I can tell the difference is only that the author is thinking of algebraic operations. This does not seem to be a different metaphor the way "function as map" and "function as transformation" are different metaphors.

Arrow Notation

The notation $f : S \to T$ means f is a function with domain S and codomain T. The is **arrow notation**, sometimes called **straight arrow notation** to distinguish it from barred arrow notation.

Think of situations where knowing the domain and codomain *is* enough to determine the function.

Example: A mathematician might write, "Let $h : \mathbb{Z} \to \mathbb{N}$ be defined by $h(n) = n^2$." This means that the domain of h is $\mathbb{Z}$, the codomain is $\mathbb{N}$ and the value at n is n^2. In this sentence the phrase $h : \mathbb{Z} \to \mathbb{N}$ would be pronounced "h from $\mathbb{Z}$ to $\mathbb{N}$".

Note that if someone writes, "Let $h : \mathbb{Z} \to \mathbb{N}$ be a function", ***they have not defined the function***. To define it you must also give some way of determining the value at each input; knowing the domain and codomain are not enough.

Extensions of the meaning of 'function"

These phrases upset some uptight types who say things like, "But a multivalued function is not a function!". A stepmother is not a mother, either. See the Handbook article on radial category.

The following phrases refer to objects that it is useful to *think of* as functions but which violate some requirement of being a function.

Multivalued function

The phrase multivalued function refers to an object that is like a function $f : S \to T$ except that for $s \in S$, $f(s)$ may denote more than one value. Multivalued functions arose in considering complex functions such as $\sqrt{z}$.

The indefinite integral is a multivalued operator.

A multivalued function $f : S \to T$ can be modeled as a function with domain S and codomain the set of all subsets of T. The two meanings are equivalent in a strong sense (naturally equivalent). Even so, it seems to me that they represent two different ways of *thinking about* multivalued functions. ("The value may be any of these things..." as opposed to "The value is this whole set of things.")

Partial function

A partial function $f : S \to T$ is just like a function except that its input may be defined on only a subset of S. This models the behavior of computer programs (algorithms): if you consider a program with one input and one output as a function, it may not be defined on some inputs because for them it runs forever (or gives an error message).

In many texts in computing science and mathematical logic, a function is by convention a partial function.

Ways of defining a function

When a math text "defines a function" it may or may not tell you what the domain or the codomain are. Sometimes, the context makes the domain or codomain clear and other times it doesn't really matter what it is. More about this in Functions: Specification and Definition.

Table

A function defined on a finite set may be given by a table; for example, the finite function defined in the examples section. The table determines the domain of the function (the domain is the set of all first coordinates of the ordered pairs in the table), but not the codomain.

Formula

A function may be defined by giving an algebraic expression (its **formula**) that determines its value. An example is the function g defined by $g(x) = x^2 + 2x - 4$.

• The expression "$g(x) = x^2 + 2x - 4$" may be called the **defining equation** of the function, and "$x^2 + 2x - 4$" its **defining expression**.

• The defining expression may be used as its name , as in "$x^2 + 2x - 4$ has one local minimum".

• In a calculus book you can assume the domain is $\mathbb{R}$. In a complex analysis text its domain could be $\mathbb{C}$, but in that text it would probably be written "$g(z) = z^2 + 2z - 4$". That is by convention.

• Two functions given by different formulas may be the same function. If you define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + 2x - 4$ and $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = (x + 1)^2 - 5$ then f and g are the ***same function***.

Algorithm

You may define a function using an **algorithm**, which to start with you can think of as a computer program that calculates the function's value at every input.

One example most everyone sees in high school or college is Newton's method for finding the root of a polynomial. You can find examples on the web of programs implementing Newton's method in Pascal, C, and Mathematica (and dozens of other languages).

If you think more about this idea, you run into subtleties:

• The program may fail or run forever on some inputs. This is true of Newton's method.

• Are two programs in different languages that "do the same thing" the "same algorithm".

• Are two programs that give the same output for each input the "same algorithm"?

"Give the same output" and "do the same thing" are not the same thing!

You can read more about this in the Wikipedia article on algorithms.

Geometric definition

The circumference function $C(r)$ could be defined using a **geometric definition** this way: "$C(r)$ is the circumference of a circle with radius r". Of course it can be given by a formula, too: $C(r) = 2\pi r$.

Be clear that ***the geometric definition is just as good a definition as the formula is***. ***It defines the function as exactly as the formula*** $C(r) = 2\pi r$ ***does***, although of course if you don't know the formula, you have to do some reasoning to figure out what $f(3)$ is (for example).

Barred arrow notation

Another naming technique is **barred arrow notation.** If E is some mathematical expression that has a definite value for each x in the domain, then you can refer to the function $x \mapsto E$ without having to give it a name.

Barred arrow notation may not be familiar to you, but it is becoming more common. Like the defining expression, it allows you to refer to a function without giving it a name, so it is a form of anonymous notation.

Example: "The function $x \mapsto x^2 + 2$ is positive for all real numbers x." Here E is the expression $x^2 + 2x$.

I could also have written "The function $x \mapsto x^2 + 2 : \mathbb{R} \to \mathbb{R}$ is always positive" using the barred arrow notation together with the **straight arrow notation**.

The straight arrow goes from domain to codomain.
The barred arrow goes from element of the domain to element of the codomain.

Since "$x \mapsto x^2 + 2$" is the name of a function, you can use it to show a value at the input, for example, $(x \mapsto x^2 + 2)(3) = 11$. This usage is not common, but it ought to be.

Use with parameters

Using the barred arrow clears up ambiguity when the defining expression has parameters in it.

Example: Let $(x \mapsto x^2 + yx + z) : \mathbb{R} \to \mathbb{R}$. This notation tells you that x is the function variable and y and z are parameters.

Of course, if you had written, "Consider the function $x^2 + ax + b$" the experienced reader will *assume you mean by convention* that x is the variable and a and b are parameters. The barred arrow notation does not depend on knowledge of conventions.

With finite functions

A variant of barred arrow notation is to define functions on finite sets element by element. For example the finite function F could be defined by: $1 \mapsto 3,\ \ 2 \mapsto 3,\ \ 3 \mapsto 2,\ \ 6 \mapsto 1$.

The name of a function

Functions may have **names**, for example "sine" or "the exponential function".

The name in English and the symbol for the function in the symbolic language may be different; for example, "sine" is the name of the sine function, but in the symbolic language it is called "sin".

Naming a function by a letter

A function may be named by a letter of some alphabet, for temporary use in that particular section of text.

This is what the Handbook calls a **local identifier**.

Examples:

• A paragraph may begin, "Let $f : \mathbb{R} \to \mathbb{R}$ be the squaring function", then the name of the function *in that paragraph* (or perhaps in that section) is f. The value of the function at 3 is $f(3)$, which is 9.

• "Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function with no derivative at 0." In this case, the letter ϕ (phi) does not refer to a *specific* function such as the squaring function, but an unspecified function that is subject to certain constraint.

• "Let $\psi(x) = x^2 - 1$ ". ψ is the letter psi. This names the function and simultaneously gives a formula for calculating its value.

Symbolic expressions that are *not* functions may also be given names. The expression "E" mentioned under barred arrow notation is an example. See Symbolic Language.

Global names

By convention in particular subfields of math, some letters are assumed by default to be the names of certain commonly used functions in that field.

Γ is the uppercase form of the Greek letter gamma

Example: An article about complex functions might refer to "the Γ function" without defining it. The author expects that the reader will know that it refers to a certain well-known function that generalizes the factorial function. But nothing stops mathematicians from using "Γ" with other meanings.

Naming a function by its value at x

It is common to refer to a function that has been named ϕ as "$\phi(x)$" (of course some other variable may be used instead of x). This is used with functions of more than one variable, too.

For functions given by formulas, this notation has the value of telling you what letter will be used for the input variable.

Examples:

- "Let $h(x)$ be a continuous function."
- "The function $\sin x$ is bounded."

This usage is very widespread, but strict writers would prefer "Let h be a continuous function" and "The sine function is bounded". I recommend the strict practice: "$\sin x$" is strictly speaking not the name of a function, but an expression denoting its value at x.

This does not mean I always *follow* the strict practice. (tsk)

Formula or equation as name of a function

A function may be referred to by using its defining expression or defining equation. This is common in calculus books.

Example: "The derivative of x^3 is always nonnegative."

Example: Very often the defining *equation* is used: "The derivative of $y = x^3$ is always nonnegative." If you analyze this example carefully, you see that it is literally nonsense. The equation $y = x^3$ is a statement. How can a statement have a derivative? Many mathematicians **Frown Fiercely** at this usage, but it is ubiquitous in math classrooms.

Anonymous notation

Using barred arrow notation ("the function $x \mapsto x^2 + 2x + 5$") or the defining expression ("the function $x^2 + 2x + 5$ ") to refer to a function are two examples of **anonymous notation** for functions. This means *no letter or word has been chosen to be a name for the function,* which is desirable if you expect to refer to it only once or twice.

Anonymous notation is used for many kinds of math objects other than functions, for example, set-builder notation, and the usual notation for matrices.

Another anonymous notation used in theoretical computing science is lambda notation, where you would refer to the same function as $\lambda x.x^2 + 2x + 5$. This usage is unfamiliar to most mathematicians outside computing science.

Notation for the value of a function

In most math texts and on this website, the value of a function ϕ at an input x is written $\phi(x)$. For example, if ϕ is the squaring function, $\phi(3) = 9$.

But there are several other common ways to write the value of a function:

- Factorial function: x! (postfix notation).
- Absolute value: $|x|$ (outfix notation).
- Addition: x + y (infix notation).
- Time derivative: $\dot{x}$ (over the x).
- Square root: $\sqrt{x}$ (to the left and over the x).

This section describes the major possibilities in some detail.

Prefix notation

An expression is in **prefix notation** if the function symbols are written on the left of the input. This may be referred to as "writing functions on the left".

This is the ***common way we write function values***. We write $f(x)$ for an arbitrary function of one variable, and we write $\sin x$, $\log\, x$ or $\log(x)$, and so on. This includes functions of more than one variable if the function is written with a letter. For example, let $f(x, y) = x^2 + 2y^2$. Then $f(3, 5) = 59$.

Note that $f(3, 5) \neq f(5, 3)$.

Parentheses around the input

The traditional math symbolic language has certain conventions about prefix notation.

- Most functions are written with the function named followed by the input name inside parentheses. This includes functions of two or more variables, as in "$f(3, 5)$".
- It is customary to omit parentheses around the argument for trig functions such as "sin" and often for log functions, so we may write "$\sin\pi$" or "$\log 2$".
- Many mathematical writers omit the parentheses in other situations too, writing "fx" instead of "f(x)".

Pascal and many other computer languages require parentheses around ***all*** arguments to functions. Mathematica requires square brackets and in fact reserves square brackets for that use.

Don't confuse multiplication
with prefix notation without parentheses.

"fx" does not mean "f times x" and "$\sin x$" does not mean "sin times x".

See Polish notation.

Infix notation

Infix notation is used only for functions of two variables. You write the name of the function ***between the variables***. Many functions denoted by symbols (as opposed to letters) are normally written this way, for example $x + y$ or $3/5$.

The expression "$x + y$" written in prefix notation would be "$+(x, y)$".

Juxtaposition

A special case of infix notation is **juxtaposition** or **concatenation**, which means writing nothing between two variables.

"Nothing" is not the same thing as "blank space".

• In standard algebraic notation, we write the product of numbers x and y as "xy". This is used only for variables represented by single *letters*, not digits: "23" does not mean 2 times 3.

• If f and g are composable functions, the composite is commonly written "gf".

Multiplication

Multiplication has many notations:

• $\times$ as in 3×5 or x × y. But this symbol means vector product when put between 3-dimensional vectors. More here.

• Juxtaposition, as in "xy", but only for single-letter variables. It does not work for digits.

• Centered dot as in "$3 \cdot 5$" or "$x \cdot y$" .

• Asterisk as in "$3 \star 5$" or "$x \star y$" (mostly in programming languages). Variables in programming languages tend to have multiletter names, and juxtaposition doesn't work with them.

• Blank space, as in "3 5" or "$x\ y$" (in Mathematica).

Notes

• Mathematica allows you to write any function of two variables between the arguments, but if its name is a string made up of letters, you have to mark it with tildes.

• Infix notation is also used for binary relations.

Postfix notation

Using **postfix notation**, you write the name of the function after its input. Most authors write functions of one variable in prefix notation, but some algebraists use postfix notation. Postfix notation may be called "writing functions on the right".

During the 1970's I wrote several papers using postfix notation. Many people complained. So I stopped doing so. On the other hand, the paper I wrote that got the most citations of my whole career was one of those papers. On the *other* other hand, at least three authors rewrote my proof...

Examples

• The expression $f(x)$ (which is in prefix notation) would be written $(x)f$ or xf in postfix notation. The version without parentheses is much more common.

• The symbol "!" denoting the factorial function is normally written in postfix notation.

• The expression $x + y$ in postfix notation is $(x, y)+$.

Polish notation

When the traditional infix notation is used for the basic operations of arithmetic, you have to use parentheses to distinguish between certain expressions. For example, $a + bc$ and $(a + b)\,c$ give different values for most choices of numbers a, b, c.

This use of parentheses is not the same as the use to enclose the argument of a function.

When binary operations are written in prefix or postfix notation, ***you don't need parentheses***. This is exhibited in the table. In the table I use $\star$ for multiplication because the traditional juxtaposition notation doesn't work for prefix and postfix notation. (Think about it).

Infix	Prefix	Postfix
$a + b \star c$	$+\, a \star b\ c$	$a\ b\ c \star +$
$(a + b) \star c$	$\star + a\ b\ c$	$a\ b + c\ \star$
$a * b + c$	$+ * a\ b\ c$	$a\ b * c\ +$
$a * (b + c)$	$*\ a + b\ c$	$a\ b\ c + *$

Prefix notation without parentheses is called **Polish notation** and postfix notation without parentheses is called **reverse Polish notation**.

Polish notation is named after the eminent Polish logician Jan Łukasiewicz, who invented the notation in the 1920's for use in logic. The terminology "reverse Polish notation" is a natural modification of this phrase and is not an ethnic slur.

The programming language Lisp uses a form of Polish notation and the languages Forth and PostScript use reverse Polish notation exclusively. *Most* computer languages use infix notation, which computer people call **algebraic notation**.

Outfix notation

A function is displayed in **outfix notation** (also called **matchfix notation**) if its symbol consists of characters (letters, digits and the like) or expressions put on ***both sides*** of the name of the input to the function (the argument). The pair of characters, one for each side, are called delimiters.

Example

• The notation (a, b) may denoted any one of several functions, discussed here. The delimiters are "(" (left) and ")" (right).

• The absolute value of a number r is denoted by $|r|$. The two delimiters are identical vertical lines.

• The greatest integer in x is sometimes denoted by $\lfloor x \rfloor$. For example, $\lfloor \pi \rfloor = 3$.

• Inner products on vector spaces may be denoted by $\langle v,\ w \rangle$.

Other notation styles

• The **bra-ket notation** for inner products is "$\langle v|w \rangle$". This combines infix and outfix notation.

• The way we write the definite integral can be seen as a fancy way of writing a function of three variables.

The integral $\int_a^b f(x)\,dx$ is a function that takes two numbers a and b and a function f and gives you a number.

If we call this function "Int" and write it in prefix notation, we could say $\text{Int}(a, b, f) = \int_a^b f(x)\,dx$. For example, $\text{Int}(0, \pi, \sin) = \int_0^\pi \sin\ x\,dx = 2$.

Functions like this whose inputs include functions are almost always called **operators**.

Note that the variable x does not appear in the list of arguments for Int. That is because it is a dummy variable.

• Any list $(a_1,\ a_2,\ a_3,\ ...)$ can be thought of as ***a function on its index set***. For example, the ordered triple $(3, 5, 0)$ can be regarded as the function whose value at 1 is 3, whose value at 2 is 5 and whose value at 3 is 0. In barred arrow notation, it may be written "$1 \mapsto 3, 2 \mapsto 5, 3 \mapsto 0$".

Note on terminology

The "fix" terminology comes from computing science, and some mathematicians also use it. Others, instead of saying, "I use postfix notation", will say, "I write my functions on the right," and so on.

Name and value

In math English, those who use prefix notation (the usual notation) would say that the value of a function f at an input c ($f(c)$ in the symbolic language) is "f of c". We pronounce $\sin x$ as "sine of x".

Some very common functions have a more complicated naming system.

Common arithmetic operations

First we look at addition, subtraction, multiplication and division.

• The **procedure** for carrying out the operation has a **name**, for example **addition**.

• The **verb** that describes carrying out the operation may be a different word. For example, we "**add** 3 and 4" to get 7.

• The **value** of the function at an input has another name. For example we say, "The **sum** of 3 and 4 is 7" (not "the addition of 3 and 4.")

• The operation is denoted by one or more **symbols**. For addition, the symbol is "+".

• The **symbol** has a name. The name of the symbol "+" is **plus**.

So if you ***add*** 3 and 4, you get 7, which you say is the ***sum*** of 3 and 4. If you write the result of the ***addition***, you write $3+4$, which you *pronounce* "3 ***plus*** 4".

The table shows these details for the common arithmetic functions.

function	*verb*	*symbol*	*symbol name*	*value*
addition	add	$+$	plus	sum
subtraction	subtract	$-$	minus	difference
multiplication	multiply	$\cdot$, $\times$ (note 1)	times	product
division	divide	$\div$ or $/$	(note 2)	quotient

Notes

1) Multiplication may also be denoted by the star symbol "$\star$", by concatenation or by a space. The expressions "$a \cdot b$" and "$a \times b$" are both pronounced "a times b" when the symbol denotes numerical multiplication or multiplication of matrices. The expression "$a \cdot b$" is pronounced "a dot b" when a and b are vectors, and "$a \times b$" is pronounced "a cross b" when a and b are three-dimensional vectors. Both symbols are used with other meanings, in some cases with other pronunciations, as well.

2) Either ***symbol*** "$/$" or "$\div$" may be called "the division symbol", but the phrase "a/b" or "$a \div b$" is ***pronounced*** "a divided by b". Note that for the other three operations in the table, the name of the symbol is also the way you pronounce it in an expression: "+" is pronounced "plus" and "$a+b$" is pronounced "a plus b".

The symbol "|" meaning divides also causes problems.

Other common operators

Composition of functions

• The composite of functions $f : A \to B$ and $g : B \to C$ may be written $(g \circ f)$ or gf. The value at an input x is most commonly written $g(f(x))$.

• When you pronounce $g \circ f$ you can say "g composed with f" or "the composite of g and f". It really means "do f then g". I have heard people say "f then g or f followed by g.

• Many writers blur the distinction between "composition" and "composite" and refer to $g \circ f$ as the "composition" of g and f. I personally hate this usage, but see how languages change.

Watch out: $gf(x)$ can mean either $(g \circ f)(x))$ or $g(x)f(x)$ in a situation where A has a multiplication. Also there is a horrendous problem with people wanting to write $f \circ g$ instead of $g \circ f$. Read what Wikipedia says about this. It is more polite than I am.

Differentiation and integration

Differentiation is an operator that takes a function to its derivative.

• The name of the ***function*** is "differentiation".

• The ***verb*** is "differentiate", but you can also say "take the derivative of".

• The ***result*** of differentiating f is called the **derivative** of f. As you probably know, there are bunches of ways of writing it. "Df", "$\frac{df}{dx}$", "f'" are some of them. If x is a function of t you can write the derivative of x with respect to t as "$\dot{x}$".

Shorter remarks about the integral: The operation is **integration**, the verb is **integrate**, the result of the operation is the **integral**.